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PERTURBATION OF THE SOLUTIONS OF A SECOND ORDER DIFFERENTIAL EQUATION WITH PERTURBATION OF THE BOUNDARY

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ABSTRACT 27705

This paper is concerned with the perturbation introduced into the solutions of a second order differential equation when the boundary is perturbed. The solution of the perturbed problem is expanded in a series in ϵ. The convergence of the series is proved for $|\epsilon| < 1/R$ (where $R>0$ is a constant). Author /23*

In reference 1, the method of reference 2 is used to analyze the perturba- /17 tion of the eigenvalues and eigenfunctions of a second order differential equation when the boundary is perturbed. In the present article, analogous methods are invoked to investigate the perturbation of the solutions.

Let us consider the following perturbed problem A_ϵ:

$$Ly_\varepsilon \equiv y''_\varepsilon + a(x)\, y'_\varepsilon + b(x)\, y_\varepsilon = f_\varepsilon(x) \tag{1}$$

with the boundary perturbation

$$y_\varepsilon(0) = 0, \quad y_\varepsilon(1+\varepsilon) = 0; \tag{2}$$

ϵ is a small parameter,

$$f_\varepsilon(x) = \sum_{n=0}^{\infty} \varepsilon^n f_n(x),$$

$f_n(x)$, $a(x)$, $b(x)$ are analytic functions.

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*Numbers in the margin indicate pagination in the original foreign text.

Note that the entire ensuing discussion carries over almost literally to the case when the coefficients $a(x)$, $b(x)$ represent power series in ϵ, i.e.,

$$a(x) = a_\varepsilon(x) = \sum_{n=0}^{\infty} \varepsilon^n a_n(x),$$

$$b(x) = b_\varepsilon(x) = \sum_{n=0}^{\infty} \varepsilon^n b_n(x);$$

$a_n(x)$, $b_n(x)$ are analytic functions.

For $\epsilon = 0$, the unperturbed problem A_0 is obtained:

$$Ly_0 \equiv y_0'' + a(x) y_0' + b(x) y_0 = f_0(x)$$

with boundary conditions $y_0(0) = 0$, $y_0(1) = 0$.

We will examine the two following cases. /18

Case 1. Let the unperturbed problem A_0 has a solution for any function $f_0(x)$, and for sufficiently small $\epsilon \neq 0$ let the solution $y_\epsilon(x)$ of the problem A_ϵ exist and be expandable in a power series in ϵ:

$$y_\varepsilon(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots + \varepsilon^n y_n + \ldots \tag{3}$$

Substituting (3) into (1), (2) and equating the coefficients of identical powers of ϵ, we obtain

$$Ly_0 \equiv y_0'' + a(x) y_0' + b(x) y_0 = f_0(x),\ y_0(0) = 0,\ y_0(1) = 0;$$

$$Ly_n \equiv y_n'' + a(x) y_n' + b(x) y_n = f_n(x),\ y_n(0) = 0,\ y_n(1) = -\sum_{l=1}^{n} \frac{y_{n-l}^{(l)}(1)}{l!}. \tag{4}$$

The solution $y_0(x)$, of course, will not be sought; to find $y_1(x)$, we consider the problem A_1:

$$Ly_1 \equiv y_1'' + a(x) y_1' + b(x) y_1 = f_1(x),\ y_1(0) = 0,\ y_1(1) = -y_0'(1).$$

The solution $y_1(x)$ of the problem A_1 has the form

$$y_1(x) = \int_0^1 G(x, s) f_1(s)\, ds + [-y_0'(1)]\, \varphi_1(x),$$

where $G(x,s)$ is a Green's function, $\psi_1(x)$ is a solution of the boundary value problem

$$Ly_1 \equiv y_1'' + a(x) y_1' + b(x) y_1 = 0, \;\; y_1(0) = 0, \;\; y_1(1) = 1.$$

Hence it follows that

$$\|y_1\| \leqslant M_0 \|f_1\| + M_1 |y_0'(1)|, \;\; \|y_1\| = \max_{0 \leqslant x \leqslant l} \{|y_1'(x)|, \; |y_1(x)|\}.$$

It can be demonstrated by induction that

$$\|y_n\| \leqslant M_0 \|f_n\| + M_1 \sum_{l=1}^{n} \left| \frac{y_{n-l}^{(l)}(1)}{l!} \right|. \tag{5}$$

Proceeding from (4) and (5), we prove the convergence of the expansion (3) (see case 2) by the method of majorants. The result is the following:

<u>Theorem 1</u>. For case 1, the solution $y_\epsilon(x)$ of the problem A_ϵ is given by the expansion (3), which converges for $|\epsilon| < 1/R$ (where R is a constant independent of ϵ).

Case 2. Now the limiting problem A_0 is not solvable for every function $f_0(x)$, and the corresponding homogeneous problem

$$Ly_1 \equiv y_1'' + a(x) y_1' + b(x) y_1 = 0, \;\; y_1(0) = 0, \;\; y_1(1) = 0$$

has a nontrivial solution $y_1(x)$, where $y_1'(0) \neq 0$, $y_1'(1) \neq 0$. As we know, the homogeneous problem associated with A_0,

$$L^* z_0 \equiv z_0'' - (a z_0)' + b z_0 = 0, \;\; z_0(0) = 0, \;\; z_0(1) = 0$$

also has only one (correct to a numerical factor) contrivial solution $z_0(x)$, where $(z_0, y_1) \neq 0$.

We will seek the solution $y_\epsilon(x)$ of the perturbed problem A_ϵ in the form /19

$$y_\epsilon(x) = \frac{c_0 y_1}{\varepsilon} + (y_0 + c_1 y_{-1}) + \varepsilon (y_1 + c_2 y_{-1}) + \dots + \varepsilon^n (y_n + c_{n+1} y_{-1}) + \dots \tag{6}$$

Substituting (6) onto (1), (2) and equating the coefficients of like powers of ϵ, we obtain

$$\begin{gathered} Ly_1 \equiv y_1'' + a(x)\, y_1' + b(x)\, y_1 = 0,\ y_{-1}(0) = 0,\ y_{-1}(1) = 0, \\ Ly_0 \equiv y_0'' + a(x)\, y_0' + b(x)\, y_0 = f_0(x),\ y_0(0) = 0,\ y_0(1) = -c_0 y_{-1}'(1), \\ Ly_n \equiv y_n'' + a(x)\, y_n' + b(x)\, y_n = f_n(x),\ y_n(0) = 0, \\ y_n(1) = -\sum_{l=0}^{n} c_{n-l} \frac{y_{-1}^{(l+1)}(1)}{(l+1)!} - \sum_{l-1}^{n} \frac{y_{n-l}^{(l)}(1)}{l!}. \end{gathered} \tag{7)-(8}$$

To determine c_0 and y_0, we investigate the problem

$$Ly_0 \equiv y_0'' + a(x)\, y_0' + b(x)\, y_0 = f_0(x),\ y_0(0) = 0,\ y_0(1) = -c_0 y_1'(1) = d. \tag{9}$$

Making use of Green's formula, we find that the following conditions stipulates its solvability:

$$(r_0, f_0) = c_0 z_0'(1)\, y_1'(1) \equiv -z_0'(1)\, d. \tag{10}$$

Hence it follows that

$$|c_0| \leqslant \frac{|(c_0, f_0)|}{|z_0'(1)\, y_{-1}'(1)|} \leqslant k_1 \| f_0 \|.$$

Knowing c_0, we find $y_0(x)$. If we require that the additional condition

$$(y_0, z_0) = \int_0^1 y_0(x)\, z_0(x)\, dx = 0 \tag{11}$$

be satisfied, the solution will be unique.

<u>Lemma</u>. The solution $y_0(x)$ of equation (9) with the condition (11) satisfies the following inequality:

$$\| y_0(x) \|_p = \| y_0''(x) \|_c + \| y_0'(x) \|_c + \| y_0(x) \|_c \leqslant M [\, \| f_0 \|_c + | y_0(1) | \,],$$

where M is a constant.

Proof. Let us examine the space P of the twice continuously differentiable functions $\{y_0(x)\}$ on $[0, 1]$, satisfying the conditions $y_0(0) = 0$ and (11), with the norm

$$\| y_0(x) \|_p = \| y_0''(x) \|_c + \| y_0'(x) \|_c + \| y_0(x) \|_c.$$

We also have the space Q of all pairs $\{f_0(x), d\}$, where $f_0(x)$ is a continuous function on $[0, 1]$, d is a real number associated with equation (10), with the norm

$$\| f_0(x), d \|_Q = \| f_0(x) \|_c + | d |$$

(Q is the direct sum of the space c $[0, 1]$ and the real line).

We determine the operator A from P in Q: to each $y_0(x) \subset P$ corresponds an Ay_0 of Q, where

$$Ay_0 = [f_0(x), d], \; f_0(x) = y_0'' + ay_0' + by_0 \equiv Ly_0, \; d = y_0(1) = -c_0 y'_{-1}(1);$$

where $f_0(x)$ and d satisfy the condition (10). The operator A is linear and /20
realizes a one-to-one mapping of P into Q. To each pair $[f_0(x), d]$ of Q corresponds a unique $y_0(x)$ of P, satisfying (9); therefore, according to Banach's theorem (ref. 3), there exists an inverse operator A^{1} with a finite norm $\|A^{-1}\| = M$, such that

$$\| y_0(x) \|_p \leqslant M \| f_0(x), d \|_Q,$$

or

$$\| y_0'' \|_c + \| y_0' \|_c + \| y_0 \|_c \leqslant M(\| f_0 \|_c + | d |) = M(\| f_0 \|_c + | y_0(1) |).$$

This proves the lemma.

Let all $c_0, C_1, \ldots, c_{n-1}$ and $y_0, y_1, \ldots, y_{n-1}$ be already known; to find c_n and y_n we consider the problem A_n:

$$Ly_n \equiv y_n'' + a(x) y_n' + b(x) y_n = f_n(f_n(x))$$

with the boundary conditions

$$y_n(0) = 0, \; y_n(1) = -\sum_{l=0}^{n} c_{n-l} \frac{y_{-1}^{(l+1)}(1)}{(l+1)!} - \sum_{l=1}^{n} \frac{y_{n-l}^{(l)}(1)}{l!}.$$

By direct analogy with the above method, we immediately find that the condition for its solvability is

$$(z_0, f_n) = z_0'(1) \left[\sum_{l=0}^{n} c_{n-l} \frac{y_1^{(l+1)}(1)}{(l+1)!} + \sum_{l=1}^{n} \frac{y_{n-l}^{(l)}(1)}{l!} \right].$$

Hence it follows that

$$|c_n| \leqslant k_1 \|f_n\| + k_2 \left[\sum_{l=1}^{n} |c_{n-l}| \left| \frac{y_{-1}^{(l+1)}(1)}{(l+1)!} \right| + \sum_{l>1}^{n} \left| \frac{y_{n-l}^{(l)}(1)}{l!} \right| \right], \tag{12}$$

the following estimate being applicable to the solution $y_n(x)$:

$$\|y_n\| \leqslant M_0 \|f_n\| + M_1 \left[\sum_{l=0}^{n} |c_{n-l}| \left| \frac{y_{-1}^{(l+1)}(1)}{(l+1)!} \right| + \sum_{l=1}^{n} \left| \frac{y_{n-l}^{(l)}(1)}{l!} \right| \right]. \tag{13}$$

For proof of the convergence of the expansion (6), we utilize the method of majorants. To formulate, in addition to (12) and (13), the required inequalities, we expand the solutions $y_{-1}(x)$, $y_n(x)$ and the coefficients of equations (7), (8) in power series in z in the neighborhood of the point x = 1; substituting these into the respective equations (7), (8) and equating coefficients of like powers of z^l, we readily obtain

$$\left|\frac{y_{-1}^{(l+2)}(1)}{(l+2)!}\right| \leqslant \sum_{k=0}^{l}\left|\frac{a^{(k)}(1)}{k!}\right|\left|\frac{y_{-1}^{(l+1-k)}(1)}{(l+1-k)!}\right| + \sum_{k=0}^{l}\left|\frac{b^{(k)}(1)}{k!}\right|\left|\frac{y_{-1}^{(l-k)}(1)}{(l-k)!}\right|,$$

$$\left|\frac{y_{n}^{(l+2)}(1)}{(l+2)!}\right| \leqslant \sum_{k=0}^{l}\left|\frac{a^{(k)}(1)}{k!}\right|\left|\frac{y_{n}^{(l+1-k)}(1)}{(l+1-k)!}\right| + \sum_{k=0}^{l}\left|\frac{b^{(k)}(1)}{k!}\right|\left|\frac{y_{n}^{(l-k)}(1)}{(l-k)!}\right| + \left|\frac{f_n^{(l)}(1)}{l!}\right|. \qquad (14)\text{-}(15)$$

Multiplying (14) b $z^{\iota+1}$ and (15) by $z^{\iota+2}$, then summing over ι from zero to infinity, we obtain 21

$$\sum_{l=-1}^{\infty}\left|\frac{y_{-1}^{(l+2)}(1)}{(l+2)!}\right| z^{l+1} \leqslant |y_1'(1)| + z\sum_{k=0}^{\infty}\left|\frac{a_{(1)}^{(k)}}{k!}\right| z^k \sum_{l-k=0}^{\infty}\left|\frac{y_{-1}^{(l+1-k)}(1)}{(l+1-k)!}\right| z^{l-k} +$$

$$+\sum_{k=0}^{\infty}\left|\frac{b_{(1)}^{(k)}}{k!}\right| z^k\left[z^2\sum_{l-k=1}^{\infty}\left|\frac{y_{-1}^{(l-k)}(1)}{(l-k)!}\right| z^{l-k-1}\right],$$

$$\sum_{l=-1}^{\infty}\left|\frac{y_{n}^{(l+2)}(1)}{(l+2)!}\right| z^{l+2} \leqslant |y_n'(1)|\, z + z\sum_{k=0}^{\infty}\left|\frac{a^{(k)}(1)}{k!}\right| z^k \sum_{l+1-k=1}^{\infty}\left|\frac{y_{n}^{(l+1-k)}(1)}{(l+1-k)!}\right| z^{l+1-k} + \qquad (16)\text{-}(17)$$

$$+ z^2\sum_{k=0}^{\infty}\left|\frac{b^{(k)}(1)}{k!}\right| z^k\left[\sum_{l-k=1}^{\infty}\left|\frac{y_{n}^{(l-k)}(1)}{(l-k)!}\right| z^{l-k} + |y_n(1)|\right] + \sum_{l=0}^{\infty}\left|\frac{f_n^{(l)}(1)}{l!}\right| z^{l+2}.$$

We now form the system of numbers $Y_n,\ \widetilde{y}_{-1}^{(l)}(1)$ & $\widetilde{y}_n(z) = \sum_{l=1}^{\infty}\frac{\widetilde{y}_n^{(l)}(1)}{l!} z^l$, , majorizing y_n, $y_{-1}^{(\iota)}(1)$, and

$$\|y_n(z)\| = \sum_{l=1}^{\infty}\left|\frac{y_n^{(l)}(1)}{l!}\right| z^l,$$

respectively, as well as $\widetilde{a}(z)$, $\widetilde{b}(z)$, f_n, and $\widetilde{c}_{n-\iota}$, majorizing

$$|a(z)| = \sum_{k=0}^{\infty}\left|\frac{a^{(k)}(1)}{k!}\right| z^k,\ |b(z)| = \sum_{k=0}^{\infty}\left|\frac{b^{(k)}(1)}{k!}\right| z^k,\ \|f_n\|,\ |c_{n-l}|,$$

so that

$$Y_0 = \|y_0\|,\ \widetilde{y}_{-1}'(1) = |y_{-1}'(1)|,\ \widetilde{y}_0(0) = |y_0(0)| = 0,\ \tau_0 = |c_0|, \qquad (18)$$

and we majorize equations (12), (13), (16), (17):

$$\tilde{c}_n = k_1\tilde{f}_n + k_2\left[\sum_{l=1}^{n}\tilde{c}_{n-l}\frac{\tilde{y}_{-1}^{l+1}(1)}{(l+1)!} + \sum_{l=1}^{n}\frac{\tilde{y}_{n-l}^{(l)}(1)}{l!}\right],$$

$$Y_n = M_0\tilde{f}_n + M_1\left[\sum_{l=0}^{n}\tilde{c}_{n-l}\frac{\tilde{y}_{-1}^{l+1}(1)}{(l+1)!} + \sum_{l=1}^{n}\frac{\tilde{y}_{n-l}^{(l)}(1)}{l!}\right],$$

$$\sum_{l=-1}^{\infty}\frac{y_{-1}^{l+2}(1)}{(l+2)!}z^{l+1} = \tilde{y}'_{-1}(1) + z\tilde{a}(z)\sum_{l-k=0}^{\infty}\frac{\tilde{y}_{-1}^{(l+1-k)}(1)}{(l+1-k)!}z^{l-k} + \qquad (19)\text{-}(22)$$

$$+ z^2\tilde{b}(z)\sum_{l-k=1}^{\infty}\frac{\tilde{y}_{-1}^{(l-k)}(1)}{(l-k)!}z^{l-k-1},$$

$$\tilde{y}_n(z) = Y_n z + z\tilde{a}(z)\tilde{y}_n(z) + z^2\tilde{b}(z)[\tilde{y}_n(z) + Y_n] + z^2\tilde{f}_n.$$

Equations (19)-(22) with the initial values (18) enable us to generate an /22 iterative process for the successive determination of Y_j, $\tilde{c}_j$, and $\tilde{y}_j(z)$. Let their values be known for $j < n-1$. From (19)-(22) we find Y_n, $\tilde{c}_n$, and $\tilde{y}_n(z)$. Comparing (19)-(22) with their analogous inequalities (12), (13), (16), (17) and proceeding from (18), we obtain

$$\|y_j\| \leqslant Y_j, \quad |c_j| \leqslant \tilde{c}_j, \quad \|y_j(z)\| \leqslant \tilde{y}_j(z).$$

Let us examine the formal power series

$$Y(z) = \sum_{n=0}^{\infty} Y_n z^n, \quad c(z) = \sum_{n=0}^{\infty}\tilde{c}_n z^n, \quad u(z) = \sum_{n=0}^{\infty}\tilde{y}_n(z) z^n, \qquad (23)$$

$$u_{-1}(z) = \sum_{l=0}^{\infty}\frac{\tilde{y}_{-1}^{(l+1)}(1)}{(l+1)!}z^l.$$

By virtue of (19)-(22) and (23), we have

$$
\begin{aligned}
c(z) &= \frac{\widetilde{c}_0 + k_1 f(z) + k_2 u(z)}{1 - k_2 [u_{-1}(z) - y'_{-1}(1)]}, \\
Y(z) &= Y_0 + M_0 f(z) + M_1 \{[c(z) u_{-1}(z) - \widetilde{c}_0 y'_{-1}(1)] + u(z)\}, \\
u_{-1}(z) &= \frac{\widetilde{y}'_{-1}(1)}{1 - [z \widetilde{a}(z) + z^2 b(z)]}, \\
u(z) &= \frac{[z + z^2 \widetilde{b}(z)] \{Y_0 + M_0 f(z) + M_1 [c(z) u_{-1}(z) - \widetilde{c}_0 \widetilde{y}'_{-1}(1)]\}}{1 - z [M_1 + M_1 z \widetilde{b}(z) + \widetilde{a}(z) + z \widetilde{b}(z)]}.
\end{aligned}
\qquad (24)\text{-}(27)
$$

For $z = 0$, these relations are satisfied with the values

$$
Y(0) = Y_0 = \|y_0\|, \quad u_{-1}(0) = \widetilde{y}'_{-1}(1),
$$
$$
\widetilde{y}_0(0) = |y_0(0)| = 0, \; c(0) = \widetilde{c}_0 = |c_0|.
$$

Writing the relations (24), (26), (27) in the respective forms

$$
\psi(c, u_1, u, z) = 0, \quad \Phi(c, u_1, u, z) = 0, \quad F(c, u_1, u, z) = 0,
$$

we immediately obtain

$$
\psi'_c\big|_{z=0} = 1, \quad \psi'_{u-1}\big|_{z=0} = \widetilde{c}_0 k_2, \quad \psi'_u\big|_{z=0} = k_2, \quad \Phi'_c\big|_{z=0} = 0,
$$
$$
\Phi'_{u_{-1}}\Big|_{z=0} = 1, \quad \Phi'_u\Big|_{z=0} = 0, \quad F'_c\Big|_{z=0} = 0, \quad F'_{u_{-1}}\Big|_{z=0} = 0, \quad F'_u\Big|_{z=0} = 1.
$$

Hence it follows that the Jacobian has a nonzero value at the point $z = 0$, i.e.,

$$
I = \frac{D(\psi, \Phi, F)}{D(c, u_{-1}, u)}\bigg|_{z=0} = \begin{vmatrix} 1 & \widetilde{c}_0 k_2 & k_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.
$$

It follows from the implicit function existence theorem that the system (24), (26), (27) defines a system of analytic functions $c(z)$, $u_{-1}(z)$, $u(z)$ in the neighborhood of the point $z = 0$. By virtue of the analyticity of $u_{-1}(z)$, $c(z)$, and $u(z)$, it follows from equation (25) that the function $Y(z)$ is also /23 analytic, and the power expansions (23) go from formal to convergent for sufficiently small z. The system of equations (24)-(27) shows that the coefficients Y_j, $\widetilde{c}_j$, and $\widetilde{y}_j(z)$ of the power expansions of these functions are interrelated by equation (19)-(22). Since these power expansions converge in a certain circular region $|z| < \gamma$, it follows that, assuming $0 < 1/R < \gamma$, we obtain in particular $V_n < a_1 R^n$, $\tilde{c}_n < a_2 R^n$ (where a_1, a_2 are positive constants). Consequently, we have proven the following theorem.

Theorem 2. For case 2, the solution $y_\epsilon(x)$ of the problem A_ϵ is given by the expansion (7), which converges for $|\epsilon| < 1/R$ (where $R > 0$ is a constant independent of ϵ).

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REFERENCES

1. Lin Tsung-ch'ih. Perturbation of the Eigenvalues and Eigenfunctions of a Second Order Differential Equation with Perturbation of the Boundary (Vozmushcheniye sobstvennykh znacheniy i sobstvennykh funktsiy differentsial'nogo uravneniya vtorogo poryadka pri vozmushchenii granitsy). Uspekhi Matematicheskikh Nauk (UMN), Vol. 18, No. 4, pp. 217-220, 1963.
2. Vishik, M. I. and L. A. Lyusternik. Solution of Certain Perturbation Problems in the Case of Matrices and Self-Adjoint and Nonself-Adjoint Differential Equations (Resheniye nekotorykh zadach o vozmushchenii v sluchaye matrits i samosopryazhennykh i nesamosoprayazhennykh differentsial'nykh uravneniy). UMN, Vol. 15, No. 3, pp. 3-80, 1960.
3. Lyusternik, L. A. and V. I. Sobolev. Elements of Functional Analysis (elementy funktsional'nogo analiza). Gostekhizdat (State Publishing House of Technical and Theoretical Literature), Moscow, 1951.

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